DYNAMIC NETWORK MODELS

NME WORKSHOP

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The phrase “temporal ERGMs,” or TERGMs, refers to all ERGMs that are dynamic.

The specific class of TERGMs that have been implemented thus far are called “separable temporal ERGMs,” or STERGMs.

In the relevant R package, we left open the possibility that we would develop more in the future.

Thus:

<table>
<thead>
<tr>
<th>Name of package</th>
<th>Cross-sectional</th>
<th>Dynamic</th>
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<tbody>
<tr>
<td>Name of function in package</td>
<td>ergm</td>
<td>tergm</td>
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<tr>
<td></td>
<td>ergm</td>
<td>stergm</td>
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</table>
ERGMs: Review

Probability of observing a graph (set of relationships) $y$ on a fixed set of nodes:

$$P(Y = y \mid \theta) = \frac{\exp(\theta' g(y))}{k(\theta)}$$

Conditional log-odds of a tie

$$\text{logit}(P(Y_{ij} = 1 \mid \text{rest of the graph})) = \log \left( \frac{P(Y_{ij} = 1 \mid \text{rest of the graph})}{P(Y_{ij} = 0 \mid \text{rest of the graph})} \right) = \theta' \partial(g(y))$$

where:

$g(y) =$ vector of network statistics

$\theta =$ vector of model parameters

$k(\theta) =$ numerator summed over all possible networks on node set $y$

$\partial(g(y)) =$ represents the change in $g(y)$ when $Y_{ij}$ is toggled from 0 to 1
STERGMs

- ERGMs are great for modeling cross-sectional network structure
- But they can only predict the *presence* of a tie; they are unable to separate the processes of *tie formation* and *dissolution*
- Why separate formation from dissolution?
STERGMs

- **Intuition**: The social forces that facilitate formation of ties are often different from those that facilitate their dissolution.

- **Interpretation**: Because of this, we want model parameters that can be interpreted in terms of ties formed and ties dissolved. (Of course we need data that can allow us to estimate these).

- **Simulation**: We want to be able to control cross-sectional network structure and relational durations separately in our disease simulations, matching both to data.
STERGMs

- E.g. if a particular type of tie is rare in the cross-section, is that because:
  - They form infrequently?
  - They form frequently, but then dissolve frequently as well?

- The classic approximation formula from epidemiology applies here as well:

  \[ \text{Prevalence} \approx \text{Incidence} \times \text{Duration} \]

  & up & up

  Formation & Inverse of dissolution

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STERGMs

Core idea:

- Y is now indexed by time
- Represent evolution from $Y_t$ to $Y_{t+1}$ as a product of two phases: one in which ties are formed and another in which they are dissolved, with each phase a draw from an ERGM.
- Thus, two formulas: a formation formula and a dissolution formula
- And, two corresponding sets of statistics
STERGMs

ERGM: Conditional log-odds of a tie existing

\[
\text{logit}(P(Y_{ij} = 1 | \text{rest of the graph})) = \theta' \partial(g(y))
\]

STERGM: Conditional log-odds of a tie forming (formation model):

\[
\text{logit} \left( P(Y_{ij,t+1} = 1 | Y_{ij,t} = 0, \text{rest of the graph}) \right) = \theta^+ \partial(g^+(y))
\]

STERGM: Conditional log-odds of a tie persisting (dissolution model):

\[
\text{logit} \left( P(Y_{ij,t+1} = 1 | Y_{ij,t} = 1, \text{rest of the graph}) \right) = \theta^- \partial(g^-(y))
\]

where:

- \( g^+(y) \) = vector of network statistics in the formation model
- \( \theta^+ \) = vector of parameters in the formation model
- \( g^-(y) \) = vector of network statistics in the dissolution model
- \( \theta^- \) = vector of parameters in the dissolution model
The model is expressed as log odds of tie equaling 1 given it equaled 1 at the last time step. This is done to make it consistent with the formation model, so all the math works out nicely. But it implies that the model, and thus the coefficients, should be interpreted in terms of effects on relational persistence. That said, people tend to think in terms of relational formation and dissolution, since relational dissolution is a more salient event than relational persistence. Thus, we often use the language of dissolution.
During simulation, two processes occur separately within a time step:

- $Y^+ = \text{network in the formation process after evolution}$
- $Y^- = \text{network in the dissolution process after evolution}$
- This is the origin of the “S” in STERGM
The statistical theory in Krivitsky and Handcock 2014:
- demonstrates a given combination of formation and dissolution model will converge to a stable equilibrium, i.e.:

Prevalence ≈ Incidence x Duration

This and other work in press provide the statistical theory for methods for estimating the two models, given certain kinds of data.
### Term = \( \sim \text{edges} \)

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Assuming time step is 1 day, what combo do you think is most common in empirical sexual networks?

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Term = ~edges

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STERGMs: Example of interpretation

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Why 2, and not 2+ in the interpretation of dissolution?
STERGMs: Data sources

1. Multiple cross-sections of complete network data
   - easy to work with
   - but rare-to-non-existent in infectious disease epi

2. One snapshot of a cross-sectional network (census, egocentric, or otherwise), plus information on relational durations
   - much more common
   - but introduces some statistical issues
Egocentric data in ERGMs and STERGMs

Repeating previous session:

- This approach entails constructing ‘artificial population’
  - Simulated population size does not need to match sample size (or even size of target population)
  - Don’t need to be able to ID whether different respondent’s partners are the same as one another, or are also respondents
  - We will go through simple examples in the tutorial today, delve back into this tomorrow
One cross-section + duration info

- Typically takes the form of
  - asking respondents about individual relationships (either with or without identifiers).
  - Often this is the $n$ most recent, or all over some time period, or some combination (e.g. up to 3 in the last year)
  - asking whether the relationship is currently ongoing
  - if it’s ongoing: asking how long it has been going on (or when it started)
  - if it’s over: asking how long it lasted (or when it started and when it ended)

- From this we want to estimate
  - the mean duration of relationships
  - perhaps additional information about the variation in those durations (overall, across categories of respondents, etc.)
One cross-section + duration info

Issues?

1. Ongoing durations are right-censored
   • can use Kaplan-Meyer or other techniques to deal with
One cross-section + duration info

- Issues?

2. Relationships are subject to length bias in their probability of being observed
   - This can also be adjusted for statistically
   - However, complex hybrid inclusion rules (e.g. most recent 3, as long as ongoing at some point in the last year) can make this complicated
In practice (and for examples in this course), we sometimes rely on an elegant approximation:

- If relation lengths are approximately exponential/geometric (a big if!), then the effects of length bias and right-censoring cancel out
- The mean amount of time that the ongoing relationships have lasted until the day of interview (relationship age) is an unbiased estimator of the uncensored mean duration of relationships
- Why?!?
Exponential/geometric durations suggests a memoryless processes – one in which the future does not depend on the past.

Imagine a fair, 6-sided die:

- What is the probability I will get a 1 on my next toss? $1/6$
- What is the probability I will get a 1 on my next toss given that my previous 1 was five tosses ago? $1/6$
- On average, how many tosses will I need before I get my first 1? 6
- On average, how many more tosses will I need before I get my next 1, given that my previous 1 was 8 tosses ago? 6

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<td>Support</td>
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<td>Probability mass function (pmf)</td>
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<tr>
<td>Cumulative distribution function (CDF)</td>
</tr>
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<td>Mean</td>
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Now, let’s imagine this fairly bizarre scenario:

- You arrive in a room where there are 100 people who have each been rolling one die; they pause when you arrive.
- You don’t know how many sides those dice have, but you know they all have the same number.
- You are not allowed to ask any information about what they’ve flipped in the past.
- The only information people will give you is: how many flips after your arrival does it take until they get their first 1?
- You are allowed to stay until all of the 100 people get their first 1, and they can inform you of the result.

Given the information provided you, how will you estimate the number of sides on the die?
Simple: when everyone tells you how many flips it takes from your arrival until their first 1, just take the mean of those numbers. Call it $m$.

Your best guess for the probability of getting a 1 per flip is $1/m$.

And your best guess for the number of sides is the reciprocal of the probability of any one outcome per flip, which is $1/1/m$, which just equals $m$ again.

Voila!
One cross-section + duration info

Retrospective relationship surveys are like this, but in reverse:

Dice:

Relationships:
If you have something approximating a memoryless process for relational duration, then an unbiased estimator for relationship length is to:

- ask people about how long their ongoing relationships have lasted up until the present
- take the mean of that number across respondents.

- RShiny app that shows this actually works
In practice, we find that the geometric distribution doesn’t often capture the distribution of relational durations overall. But, if you divide the relationships into 2+ types, it can do a reasonable job within type. Especially if you remove any 1-time contacts and model them separately (for populations where they are common). In our applied models (and in EpiModelHIV) we have three types. Remember: most commonly used versions of DCMs model pretty much everything as a memoryless process, so approximating one aspect of our model that way is well within common practice.
When we pass our data into EpiModel as cross-sectional structure + durations, the algorithm is going to:

- Calculate the dissolution coefficients first using data on duration
- Then estimate the formation model condition on the dissolution model, using data on cross-sectional network structure
One cross-section + duration info

- Mostly this will happen behind the scenes, but to get a flavor:

\[
\text{logit} \left( P(Y_{ij,t+1} = 1| Y_{ij,t} = 1, \text{rest of the graph}) \right) = \theta' \partial(g^-(y))
\]

- For the \sim \text{edges} model, with mean duration = 90 time steps:

\[
\ln \left( \frac{P(\text{tie persists})}{P(\text{tie dissolves})} \right) = \theta' \partial(g^-(y))
\]

\[
\ln \left( \frac{P(\text{tie persists})}{P(\text{tie dissolves})} \right) = \theta
\]

\[
\ln \left( \frac{P(\text{tie persists})}{1/90} \right) = \theta
\]

\[
\ln \left( \frac{90 - 1}{90} \right) = \theta
\]

\[
\ln(90 - 1) = \theta
\]

\[
4.49 = \theta
\]
So dissolution can be solved analytically

Then we want to condition the formation model on the dissolution model

In R, the standard notation for indicating the parameters of a model that are to be fixed and conditioned on, rather than estimated, is with:

```
~offset(FixedParameter)
```
To EpiModel.....
The "separable" part of STERGMS means that within a time step, formation and dissolution are independent.

But this does not mean that they must be independent across time steps.

Imagine this model:

- formation = ~edges + degree(2:10)
- dissolution = ~edges

with increasingly negative parameters on the degree terms.

i.e. there is some underlying tendency for relational formation to occur, which is considerably reduced with each pre-existing tie that the each actor involved is already in.

In other words, there is a strong prohibition against being in multiple simultaneous romantic relationships.

However, dissolution is fully independent---all existing relationships have the same underlying dissolution probability at every time step.
Imagine that Chris and Pat are in a relationship at time $t$. During the step between $t$ and $t+1$, whether they acquire a new partner does not depend on whether they break up and vice versa. Let us assume that they do break up during this time. Now, during the time period between $t+1$ and $t+2$: whether or not they each form new partnership is dependent on whether they are still together are time $t+1$, and that in turn depends on whether they broke up between $t$ and $t+1$. 
The simple implication of this is that in this framework, formation and dissolution can be dependent, but that dependence occurs in subsequent time steps, not simultaneously.

Note that a time step here is arbitrary, and left to the user to define. One reason to select a smaller time interval is that it makes this assumption more justifiable.

I.e. with a time step of 1 month, then if I start a new relationship today, the earliest I can break up with my first partner as a direct result of that new partnership is in one month.

If my time step is a day, then it is in 1 day

The latter is likely much more reasonable.

The tradeoff is that a shorter time interval means longer computation time for both model estimation and simulation

At the limit, this can in practice approximate a continuous-time model---the only issue is computational limitations.